MID-SEMESTER EXAMINATION
B. MATH III YEAR, II SEMESTER February 2017
ANALYSIS IV.

1. Let (X,d) and (Y,ρ) be metric spaces. On $X\times Y$, consider $r((x_1,y_1),(x_2,y_2)) = \sqrt{d(x_1,x_2)^2+\rho(y_1,y_2)^2}$. Show that r is a metric.

Solution: Staraightforward verification.

2. Let X be a metric spac and A be a compact set in X with more than two points. Show that the diameter of A is $d(a,b)$ for some $a,b \in A$.

Solution: By definition, the diameter is $d(A) = \sup\{d(x,y): \ x,y \in A\}$. Again, there exist (x_n) and (y_n) in A, such that $d(x_n,y_n) \to d(A)$ as $n \to \infty$. From the compactness of A, we have subsequences (x_{n_k}) and (y_{m_k}), each having limit points, say x and y in A. Then $d(A) = d(x,y)$.

3. Show that the set of all irrationals with sup metric is separable.

Solution: We need a countable set of irrations, which is dense in the set of irrationals. Let $\{q_1, q_2, \cdots,\}$ be the enummeration of rationals. Choose irrational $r_{j,k}$ that lies in $(q_j - 1/k, q_j + 1/k)$. Then these irrationals are dense in the given set.

4. If (X,d) is a complete metric space and A is totally bounded in X, then $\overline{A}$ is compact in X.

Solution: This is available in standard books; see Simmons, for example.

5. Let (X,d) be a compact metric space. Let (U_α) be family of open sets in X whose union is X. Show that there is a countable sub-family of the given family, that covers X.

Solution: Standard.

6. Show that $G = \{f \in C[0,1]: \ \sup|f(t)| \leq 1\}$ is not an equicontinuous set.

Solution: Define $h_n : [-1,1] \to \mathbb{R}$ by

$$h_n(x) = \begin{cases} -1, & \text{if } x < -1/n \\ nx, & \text{if } -1/n \leq x \leq 1/n \\ 1 & \text{if } x > 1/n. \end{cases} \tag{0.1}$$

The given family is not equicontinuous at 0. Equicontinuity at 0 means that, for $\epsilon > 0$, there exists δ so that $|h_n(x) - 0| < \epsilon$ whenever $|x| < \delta$, for all n. From this it follows that if x_k converges to 0, then $h_n(x_k)$ converges to 0 as k goes to ∞, uniformly in n. But the given function does not satisfy this. For example, $x_k = \frac{1}{\sqrt{k}}$ goes to 0. But $h_k(x_k) = 1$ for all k.

7. Give examples of metric spaces X and Y and a uniformly continuous function $f : X \to Y$ such that for no $0 < c < 1$, the function satisifies $d_1(f(x), f(y)) \leq c d_2(x, y)$.

Solution: Consider the function $f(x) = 4x(1-x)$ from $[0, 1]$ to itself. This is uniformly continuous. There cannot be a c as in the question. If there is, then the iterates $f^n(x)$ converges to a unique point. But this is not true. Banach fixed point theorem gives the result.

8. Let f be a continuous function on $[0, 1]$ with $f(0) = 0$. Show that there is a polynomial p with $p(0) = 0$ such that $\sup |f(t) - p(t)| \leq \epsilon$.

Solution: Follow Weirstrass theorem's proof.